MATHEMATICS OF COMPUTATION
VOLUME 59, NUMBER 199
JULY 1992, PAGES 203-211

IMPROVEMENT OF NAKAMULA'S UPPER BOUND FOR THE ABSOLUTE DISCRIMINANT OF A SEXTIC NUMBER FIELD WITH TWO REAL CONJUGATES

R. J. STROEKER

ABSTRACT. Let $\mathbf{K}$ be the compositum of a real quadratic number field $\mathbf{K}_2$ and a complex cubic number field $\mathbf{K}_3$. Further, let ε be a unit of $\mathbf{K}$ which is also a relative unit with respect to $\mathbf{K}/\mathbf{K}_2$ and $\mathbf{K}/\mathbf{K}_3$. The absolute discriminant of this non-Galois sextic number field $\mathbf{K}$ is estimated from above by a simple, strictly increasing, polynomial function of ε. This estimate, which can be used to determine a generator for the cyclic group of relative units, substantially improves a similar bound due to Nakamula. The method employed makes nontrivial use of computer algebra techniques.

1. INTRODUCTION

In his beautiful paper [2], Nakamula shows great arithmetic skills in the way he calculates fundamental units and class numbers of number fields of absolute degree 6 over $\mathbf{Q}$. In the paper cited, Nakamula considers sextic fields $\mathbf{K}$ with a real quadratic subfield $\mathbf{K}_2$ and a complex cubic subfield $\mathbf{K}_3$. Or rather, take any quadratic extension $\mathbf{K}_2$ of $\mathbf{Q}$ of discriminant $d_2 > 0$ and any cubic extension $\mathbf{K}_3$ of $\mathbf{Q}$ of discriminant $d_3 < 0$; then the compositum $\mathbf{K} := \mathbf{K}_2 \cdot \mathbf{K}_3$ is a non-Galois sextic field of positive absolute discriminant D. Consider $\mathbf{K}$ to be embedded in the reals $\mathbf{R}$.

It is the unit structure of $\mathbf{K}$ that interests us. As the torsion subgroup of the unit group is trivial, it is sufficient to consider only the group E of positive units of $\mathbf{K}$. Let H be its subgroup of relative units with respect to $\mathbf{K}_2$ and $\mathbf{K}_3$, that is,

$$H := \{\varepsilon \in E \,|\, \mathrm{Norm}_{\mathbf{K}/\mathbf{K}_2}(\varepsilon) = \mathrm{Norm}_{\mathbf{K}/\mathbf{K}_3}(\varepsilon) = 1\}. \tag{1}$$

Both free unit groups E_2 and E_3 of positive units of $\mathbf{K}_2$ and $\mathbf{K}_3$, respectively, have rank 1, and E has rank 3, by Dirichlet's unit theorem. Consequently, H is an infinite cyclic group. One of the main points of Nakamula's paper is to give an effective way of calculating the three generators of the free unit group E, and he succeeds in doing this by expressing the generators of E in terms of the generators of E_2, E_3, and H.

Assuming that generators for E_2 and E_3 have been found, we concentrate on H.

Received by the editor January 12, 1991 and, in revised form, July 3, 1991.
1991 *Mathematics Subject Classification*. Primary 11R21, 11R29, 11-04.

Let $\varepsilon \in H$, $\varepsilon > 1$, be given. To ensure that ε generates H, is is sufficient to check that a relation of the form

(2) $$\varepsilon = \xi^n \quad \text{for a } \xi \in H \text{ and } n \in \mathbf{N} \text{ with } n \geq 2$$

is impossible. In order to do this, suppose that a strictly increasing function $T\colon \mathbf{R}^+ \to \mathbf{R}^+$ exists such that for each $\xi \in H$ the inequality $D(\xi) < T(\xi)$ holds, where $D(\xi)$ is the discriminant of ξ with respect to $\mathbf{K}/\mathbf{Q}$. If the inverse T^{-1} of T can be explicitly evaluated at each relevant value, then a useful upper bound for the exponent n in (2) can be obtained as follows. Let D be the absolute discriminant of $\mathbf{K}$. As $\varepsilon > 1$, it follows from (2) that

$$0 < D \leq D(\xi) = D(\sqrt[n]{\varepsilon}) < T(\sqrt[n]{\varepsilon}),$$

which implies $n < \log\varepsilon / \log T^{-1}(D)$, provided $D > T(1)$. In Lemma 2 of [2] the author uses the function

$$T(x) := 16((x + \tfrac{9}{7})^7 - 290)^2,$$

to prove (see Theorem 1 of [2])

Theorem 1 (Nakamula). *Let ε and n be as in* (2). *Then*

$$n < B(\varepsilon) := \frac{\log\varepsilon}{\log\left(\sqrt[7]{\sqrt{D}/4 + 290} - 9/7\right)}.$$

Although his result is correct, Nakamula's proof contains some misprints (see §3) and besides, the argument is not sufficiently transparent to enable the reader to make the necessary adjustments. The object of the present paper is to give a different proof, making extensive use of computer algebra methods—in fact the Macintosh SE implementation of Maple, version 4.2.1 (see [1]) is used—of the following theorem:

Theorem 2. *Let ε and n be as in* (2). *Then*

$$n < \widetilde{B}(\varepsilon) := \frac{\log\varepsilon}{\log\left(\sqrt[7]{\sqrt{D}/4 + 76} - 6/7\right)}.$$

It is an easy exercise to show that the upper bound $\widetilde{B}(\varepsilon)$ improves Nakamula's bound $B(\varepsilon)$. Indeed, on putting

$$\lambda := \sqrt[7]{\sqrt{D}/4 + 76} \quad \text{and} \quad \mu := \sqrt[7]{\sqrt{D}/4 + 290},$$

it follows that $\lambda > 2$ and $\mu > \frac{9}{4}$, as the smallest possible discriminant D equals 66125 (see [2, p. 244]). Application of these inequalities yields

$$214 = \mu^7 - \lambda^7 = (\mu - \lambda)\sum_{i=0}^{6} \mu^i\lambda^{6-i} > (\mu - \lambda)\frac{2685817}{4096},$$

and hence $\mu - \lambda < \frac{3}{7}$, from which the assertion immediately follows.

The real improvement of $\widetilde{B}(\varepsilon)$ over $B(\varepsilon)$ lies in the reduction of the fraction $\frac{9}{7}$ to $\frac{6}{7}$. How substantial this improvement can be for relatively small discriminants is clear from the table in §5. For large discriminants it is not immediately clear whether $\widetilde{B}(\varepsilon)$ always induces a better *integer* bound than $B(\varepsilon)$.

2. The discriminant of a relative unit

Let ε be a relative unit, that is, $\varepsilon \in H$ as in (1). If the conjugates of ε with respect to $\mathbf{K}/\mathbf{K}_3$ are ε and ε', then $\varepsilon\varepsilon' = 1$, and hence $\varepsilon' = \varepsilon^{-1}$. This shows that we may assume $\varepsilon > 1$, which we shall do from now on. With respect to $\mathbf{K}/\mathbf{K}_2$, let ε, ε'', and ε''' be the conjugates of ε, so that $\varepsilon\varepsilon''\varepsilon''' = 1$. As $\mathbf{K}_3$ is not totally real, ε'' and ε''' are complex conjugates. All this implies that the field conjugates of ε are

$$\varepsilon,\ \frac{1}{\varepsilon},\ \sqrt{\varepsilon}\exp(i\phi),\ \frac{1}{\sqrt{\varepsilon}}\exp(i\phi),\ \sqrt{\varepsilon}\exp(-i\phi),\ \frac{1}{\sqrt{\varepsilon}}\exp(-i\phi), \tag{3}$$

for some ϕ with $0 < \phi < \pi$. Further, let $\theta := \mathrm{Tr}_{\mathbf{K}/\mathbf{K}_3}(\varepsilon)$, $\theta'' := \mathrm{Tr}_{\mathbf{K}/\mathbf{K}_3}(\varepsilon'')$, and let θ''' be the complex conjugate of θ''. Then

$$\begin{aligned}&(x^2 - \theta x + 1)(x^2 - \theta'' x + 1)(x^2 - \theta''' x + 1)\\ &\quad = x^6 - sx^5 + tx^4 - ux^3 + tx^2 - sx + 1,\end{aligned} \tag{4}$$

with s, t, $u \in \mathbf{Z}$, is the form of the minimal polynomial of ε over $\mathbf{Q}$. On setting

$$a := \sqrt{\varepsilon} + 1/\sqrt{\varepsilon}, \tag{5}$$

we deduce from (3) and (4) that

$$\begin{aligned}s &= \theta + 2a\cos\phi,\\ t &= 1 + \theta(1 + s - \theta) + a^{-2}(s - \theta)^2.\end{aligned} \tag{6}$$

A relation between s, t, and u is quickly established as follows. On putting $\alpha := \mathrm{Tr}_{\mathbf{K}/\mathbf{K}_2}(\varepsilon)$ and $\alpha' := \mathrm{Tr}_{\mathbf{K}/\mathbf{K}_2}(\varepsilon')$, we find that the minimal polynomial of ε is also given by

$$(x^3 - \alpha x^2 + \alpha' x - 1)(x^3 - \alpha' x^2 + \alpha x - 1),$$

so that

$$s = \alpha + \alpha', \quad t = \alpha + \alpha' + \alpha\alpha', \quad u = 2 + \alpha^2 + \alpha'^2,$$

from which we immediately deduce the relation $u = s^2 + 2s - 2t + 2$. Hence, the minimal polynomial of ε is

$$x^6 - sx^5 + tx^4 - (s^2 + 2s - 2t + 2)x^3 + tx^2 - sx + 1, \qquad s, t \in \mathbf{Z}. \tag{7}$$

Finally, it is now rather easy to express the discriminant $D(\varepsilon)$ with respect to $\mathbf{K}/\mathbf{Q}$ in terms of ε and ϕ. If

$$F(x) := (a^3 - 3a - x)^2(a - x)^3(a + x)(4 - x^2), \qquad x \in \mathbf{R}, \tag{8}$$

then we have

$$\sqrt{D(\varepsilon)} = (\varepsilon - \varepsilon^{-1})(\alpha - 2)F(2\cos\phi). \tag{9}$$

Note that $D(\varepsilon) > 0$, as $\mathbf{K} = \mathbf{Q}(\varepsilon)$ has an even number of complex conjugate pairs. Because of (9), it goes without saying that we are only interested in the range $(-2, 2)$ of arguments of the function F.

If for all $a > 2$, $U_F(a)$ is an upper bound for the function F on $(-2, 2)$, then

$$\sqrt{D(\varepsilon)} \leq aU_F(a)\sqrt{(a^2-4)^3}. \tag{10}$$

Hence, in order to find an upper bound for the discriminant $D(\varepsilon)$ in terms of ε or a, it is sufficient to find one for the function F on the interval $(-2, 2)$.

3. Upper bound for the function F

The function F is a polynomial in x of degree 8, the coefficients of which are polynomials in a. Hence, solving $F'(x) = 0$ analytically—in order to find stationary points—is either trivial (which is not the case here) or impossible, the more so because a parameter a is involved. Now Nakamula in the paper cited gets around this problem by estimating the stationary point in terms of a. However, the phrase "after a tedious calculation" does not give any insight into what really happens. Moreover, there are a few misprints in the proof. To be more precise, on p. 231 of [2], the constant term in the definition of A_3 should be -104 instead of -108, and the coefficient of γ^{-3} in the closing line should be $+1280$ instead of -1280. We shall proceed in a different way.

First of all, as $a > 2$ and hence $a^3 - 3a > 2$, the function F does not vanish on the interval $(-2, 2)$. Further, as $F(0) > 0$, F must have a unique and positive maximum $M_F(a)$ at $x = x(a)$ on $(-2, 2)$ for all $a > 2$.

The general approach we plan to adopt, and which ultimately leads to an upper bound for M_F, may be described as follows.

Main procedure. For every x-interval $I \subset (-2, 2)$ containing $x(a)$ and on which F is a concave function, the graph of F lies entirely below the tangent to the graph at any point with x-coordinate belonging to I. Now take two points in I, one to the left and one to the right of $x(a)$. Then the tangents to the graph of F at the corresponding points on the graph intersect in a point with x-coordinate $\geq M_F$.

So we have to determine a suitable subinterval I of $(-2, 2)$ and suitable points sufficiently close to $x(a)$ to the left and right of $x(a)$ to make the process work. By inspection, $F'(-1) > 0$ and $F'(-\frac{2}{a}) < 0$ (see the list of F'-values in [3]), so that $x(a)$ belongs to the interval $(-1, -\frac{2}{a})$ for all $a > 2$. However, as $\lim_{a\to\infty} F'(-1)/a^{10} = 2$ and $\lim_{a\to\infty} F'(-\frac{2}{a})/a^{10} = 0$, the value -1 is a rather bad choice for large values of a; the left endpoint $-\frac{4}{a}$ apparently is a better choice. Unfortunately, for values of a close to 2, the function F is not concave on the corresponding interval $(-\frac{4}{a}, -\frac{2}{a})$. For instance, if $a = 2$ and $-2 < x < -1 - \frac{1}{7}\sqrt{21}$, then $F''(x) > 0$. As the sign of F'' cannot be determined by mere inspection, we proceed by attempting to locate all real zeros of this polynomial function. As it turns out, all six zeros of F'' are real, amongst which the simple zero $x = a$. Now, by determining all sign changes, we shall know approximately where F'' takes negative values. Let $\overline{F}$ be the polynomial of degree 5 over $\mathbf{R}$ with

$$\overline{F}(x) = \frac{F''(x)}{a - x} \quad \text{for } x \neq a.$$

Clearly, the leading coefficient of $\overline{F}(x)$ is -1, and our table of signs indicates that $\overline{F}$ does not change sign on the intervals $(-1, -\frac{2}{a})$ and $(-\frac{3}{a}, -\frac{2}{a})$. Indeed,

The sign of the function $\overline{F}$
$s(x) :=$ sign of $\overline{F}(x)$; $p(a) := \frac{3}{4}a^3 - a^2 + a - 2$

x	$-\infty <$	-1 and $-\frac{3}{a}$	$-\frac{2}{a}$	$\frac{a}{2}$	a	$p(a)$	$< \infty$
$s(x)$	$+$	$-$	$-$	$+$	$-$	$+$	$-$

the position of every one of the five zeros of $\overline{F}$ is accounted for outside these intervals. Note that

$$\max\left(-1, -\frac{3}{a}\right) < -\frac{2}{a} < \frac{a}{2} < a < p(a) = \frac{3}{4}a^3 - a^2 + a - 2$$

for $a > 2$.

It is not easy to prove that the sign of $\overline{F}(x)$ at each of the six given x-values of the table is as indicated. The reason is that for certain rational functions of a it needs to be established that no sign change occurs as a ranges through $(2, \infty)$. Here the power of symbolic computation is needed for the first time. What we do is this. We simply substitute in $\overline{F}(x)$ for x the relevant value expressed in terms of a, followed by the substitution of $a = b + 2$. Of the resulting rational function of b, the numerator decides its sign, as the denominator is trivially positive for $b > 0$. As it turns out, all nonzero coefficients are of equal sign, so that no sign change occurs in the range $b > 0$. The explicit polynomials are printed in [3].

The following lemma now easily follows.

Lemma. *Let F, defined as in* (8), *attain its unique maximum at $x(a) \in (-2, 2)$. Further, let $I_3(a) := (-\frac{3}{a}, -\frac{2}{a})$ and $I_4(a) := (-\frac{4}{a}, -\frac{2}{a})$. Then*

(i) *$x(a) \in I_3(a)$ and F is concave on $I_3(a)$ if $2 < a \leq 4$,*

(ii) *$x(a) \in I_4(a)$ and F is concave on $I_4(a)$ if $a > 4$.*

The concavity of the function F in both cases has been established above. To show that $x(a)$ belongs to the relevant interval, it is sufficient to prove that $F'(-\frac{3}{a}) > 0$ if $2 < a \leq 4$ and $F'(-\frac{4}{a}) > 0$ if $a > 4$, as we already know that $F'(-\frac{2}{a}) < 0$. These assertions follow from the F'-values as given in [3]. Note that $F'(-\frac{3}{a}) > 0$ in the range $a > 2$ only if $a^8 - 32a^6 + 174a^4 - 378a^2 + 324 < 0$, and the positive real zeros of this polynomial are 1.61662706 and 5.07997821, approximately.

To continue the main process, select $I = I_3(a)$ or $I = I_4(a)$, and calculate an upper bound $U_F(a)$ for F on I by intersecting the tangents to the graph of F at the points corresponding to the endpoints of I. It is clear that $U_F(a)$ is a rational function of a. Note that, owing to the choice of endpoints, $U_F(a)$ is in fact a rational function of a^2. In the next section we shall construct an upper bound for $\sqrt{D(\varepsilon)}$ by means of (10), for both choices of I.

4. Symbolic computations

In this section we shall finalize the proof of Theorem 2 by manipulating polynomials with large rational coefficients and of rather high degree. Explicit information, including Maple programs, is provided in [3].

The right-hand side of inequality (10) is not a rational function of a. To restore rationality, we substitute for a the expression $c+c^{-1}$; in fact we have $c := \sqrt{\varepsilon}$, and then $a = c + c^{-1}$ by (5). Note that $c > 1$ by assumption. Next we define polynomials P, $Q \in \mathbf{Z}[c]$ by

$$\frac{P(c)}{Q(c)} := U_F(c+c^{-1})(c+c^{-1})(c-c^{-1})^3,$$

where the fraction is normalized. Then (10) yields

$$\sqrt{D(\varepsilon)} \leq \frac{P(c)}{Q(c)}.$$

As $U_F(a)$ is a rational function in the variable a^2, both $P(c)$ and $Q(c)$ are polynomials in c^2. For both choices of interval I (see Lemma), the denominator $Q(c)$ has nonnegative integer coefficients, so that by the Euclidean algorithm,

$$\frac{P(c)}{Q(c)} = M(c) + \frac{R(c)}{Q(c)},$$

with M, $R \in \mathbf{Z}[c]$ and $\deg(R) < \deg(Q)$; it follows that for all $c > 1$

$$\frac{P(c)}{Q(c)} \leq M(c) + d_1 \leq a_M(c^2+d_2)^{\deg(M)/2} + d_3, \tag{11}$$

for suitable constants d_1, d_2, and d_3. Here, a_M denotes the leading coefficient of $M(c)$. Our object is now to choose both d_2 and d_3 minimal, subject to inequality (11).

First case: $I_3(a)$. In case $I = I_3(a) = (-\frac{3}{a}, -\frac{2}{a})$, we have the polynomials

$$P(c) = 4c^{74} + 150c^{72} + 2370c^{70} + 20718c^{68} + \cdots,$$
$$Q(c) = c^{60} + 32c^{58} + 420c^{56} + 2996c^{54} + \cdots.$$

Next consider (see (11))

$$V(c) := -P(c) + Q(c)(4(c^2+d_2)^7 + d_3) = (28d_2 - 22)c^{72} + \cdots. \tag{12}$$

The best possible choice for d_2 is such that the leading coefficient of $V(c)$ vanishes, i.e., $d_2 = \frac{11}{14}$. Next substitute $c = 1 + b$ in the resulting expression for $V(c)$ and choose d_3 minimal and such that all coefficients of $V(1+b)$ as polynomials in b are nonnegative. The optimal choice is now

$$d_3 = -\frac{6103515625}{26353376} = -231.602\ldots,$$

which is obtained by setting $V(1) = 0$, corresponding to $b = 0$. Then

$$V(1+b) = \frac{461}{7}b^{70} + 4610b^{69} + \frac{15805277}{98}b^{68} + \frac{184087458}{49}b^{67} + \cdots,$$

the coefficients of which can have numerators as large as 10^{30}. In [3] all coefficients of this polynomial are given.

The final inequality is

$$\frac{P(c)}{Q(c)} \leq 4\left(c^2 + \frac{11}{14}\right)^7 - 231, \tag{13}$$

which is true for all c with $1 < c \le 2+\sqrt{3}$, as $2 < a \le 4$.

Second case: $I_4(a)$. Now we have $I = I_4(a) = (-\frac{4}{a}, -\frac{2}{a})$, and the corresponding polynomials are

$$P(c) = 4c^{70} + 172c^{68} + 3084c^{66} + 30308c^{64} + \cdots,$$
$$Q(c) = c^{56} + 37c^{54} + 553c^{52} + 4453c^{50} + \cdots.$$

As before (see (12)),

$$V(c) := -P(c) + Q(c)(4(c^2+d_2)^7 + d_3) = (28d_2 - 24)c^{68} + \cdots,$$

and here the best choice for d_2 is $\frac{6}{7}$. Again, substitute $c = 1+b$ in the resulting expression for $V(c)$ and choose d_3 such that all coefficients of $V(1+b)$ are nonnegative. The optimal choice is

$$d_3 = -\frac{250994068}{823543} = -304.773\ldots,$$

which again is obtained by setting $V(1) = 0$. Then

$$V(1+b) = \frac{544}{7}b^{66} + \frac{35904}{7}b^{65} + \frac{8322392}{49}b^{64} + \frac{184124928}{49}b^{63} + \cdots.$$

As in the first case, the coefficients can have very large numerators. See [3] for full information.

The result is that

$$\frac{P(c)}{Q(c)} \le 4\left(c^2 + \frac{6}{7}\right)^7 - 304, \tag{14}$$

for all $c > 2+\sqrt{3}$, because here $a > 4$.

It remains to combine both inequalities (13) and (14) and create a single inequality valid for all values of $a > 2$. This is a straightforward matter. From

$$\begin{aligned}\left(c^2+\frac{6}{7}\right)^7 - \left(c^2+\frac{11}{14}\right)^7 &= \sum_{n=0}^{6}\binom{7}{n}\left(c^2+\frac{11}{14}\right)^n\left(\frac{1}{14}\right)^{7-n}\\ &> \sum_{n=0}^{6}\binom{7}{n}\left(1+\frac{11}{14}\right)^n\left(\frac{1}{14}\right)^{7-n}\\ &= \left(\frac{13}{7}\right)^7 - \left(\frac{25}{14}\right)^7 > \frac{73}{4},\end{aligned}$$

it follows that for all $c > 1$

$$4\left(c^2+\frac{6}{7}\right)^7 - 304 > 4\left(c^2+\frac{11}{14}\right)^7 - 231.$$

Hence, for all $a > 2$

$$D \le D(\varepsilon) \le 16\left(\left(c^2+\frac{6}{7}\right)^7 - 76\right)^2,$$

and Theorem 2 immediately follows.

5. Comparison of bounds

In this final section we shall give examples to demonstrate the extent to which the bound $\widetilde{B}(\varepsilon)$ is an improvement of $B(\varepsilon)$, the one given by Nakamula in [2].

Comparing $B(\varepsilon)$ and $\tilde{B}(\varepsilon)$

ε has minimal polynomial (7) with parameters s and t,

1.2(3) is short for 1.2×10^3

D, D_2 and D_3 are the discriminants of $\mathbf{K}$, $\mathbf{K}_2$ and $\mathbf{K}_3$

Entry n_i in column 1 corresponds to Table n, line i,

and (*) corresponds to Example 7.2.(iii) of [2]

[2]	s	t	D	D_2	$-D_3$	ε	$B(\varepsilon)$	$\tilde{B}(\varepsilon)$
3_7	−1	−9	1760913	33	231	3.4(0)	6.2	3.0
3_3	−1	−8	390224	29	116	3.3(0)	11.9	4.3
2_3	−1	−6	7260624	21	588	2.9(0)	3.3	2.0
3_2	−1	−4	140608	13	104	2.4(0)	16.9	4.4
3_{25}	0	−2	1782272	8	472	2.2(0)	4.0	2.0
3_9	1	−14	3631696	61	244	5.1(0)	6.3	3.4
3_1	1	−6	219501	29	87	3.8(0)	18.9	5.7
3_{12}	1	−3	1105425	17	255	3.2(0)	7.0	3.1
2_1	1	0	450000	5	300	2.1(0)	7.0	2.6
3_{23}	2	−37	7744000	40	440	8.0(0)	6.3	3.8
2_7	2	−9	4320000	12	300	5.0(0)	5.8	3.3
3_4	2	−5	184832	8	152	4.3(0)	22.9	6.6
3_5	5	−2	2382032	53	212	6.3(0)	8.3	4.2
3_{20}	7	−8	20720464	109	436	8.9(0)	5.1	3.4
3_{16}	10	−78	12986073	113	339	1.6(1)	7.3	4.6
3_{19}	19	96	9528128	53	424	1.3(1)	7.2	4.5
3_6	22	−193	1120581	21	231	3.0(1)	20.4	9.2
3_{10}	22	−65	793117	13	247	2.6(1)	22.8	9.6
3_{18}	22	6	23142177	137	411	2.3(1)	7.1	4.8
3_{11}	22	63	325125	5	255	2.0(1)	33.2	11.3
3_{21}	23	124	968000	5	440	1.7(1)	18.1	8.0
3_8	25	162	4108797	77	231	1.6(1)	10.3	5.8
2_{10}	28	191	69574032	33	1452	1.9(1)	5.3	3.9
3_{24}	30	−401	8339441	41	451	4.1(1)	11.0	6.7
(*)	38	319	66125	5	23	2.8(1)	122.8	21.2
2_2	42	375	559872	12	108	3.1(1)	28.6	11.2
3_{13}	49	458	5527125	85	255	3.8(1)	12.2	7.1
3_{14}	81	−486	11655261	109	327	8.8(1)	12.1	7.6
3_{15}	86	−65	561125	5	335	8.8(1)	37.2	14.6
3_{17}	110	−2561	11279504	89	356	1.3(2)	13.3	8.4
2_4	141	−3090	2278125	5	675	1.6(2)	23.1	11.8
2_5	142	2735	12778713	17	867	1.2(2)	12.6	8.0
2_6	158	2751	66854673	57	1083	1.4(2)	8.9	6.5
3_{22}	266	−9893	1548800	8	440	3.0(2)	30.1	14.4
2_9	1581	−111810	36756909	21	1323	1.6(3)	15.1	10.6
2_8	1666	59015	21600000	60	300	1.6(3)	17.0	11.4

It seems fair to choose examples similar to those provided there. In fact, we have chosen exactly the same (s, t)-values as those offered in the examples and in Tables 2 and 3 of the paper cited. The first column of the table gives a reference to Nakamula's tables and examples. We noticed a couple of misprints in these tables: the first entry of the second line of Table 2 should be 80 instead of 60 and the third entry of line 4 of Table 3 should be -8 and not 8.

ACKNOWLEDGMENT

The author gratefully acknowledges the referee's clarification of Nakamula's argument, pointing out the actual misprints on p. 231 of [2].

BIBLIOGRAPHY

1. Macintosh Maple 4.2.1, Symbolic Computation Group, Department of Computer Science, Univ. of Waterloo, 1990.
2. Ken Nakamula, *Class number calculation of a sextic field from the elliptic unit*, Acta Arith. **45** (1985), 230–247.
3. R. J. Stroeker, *Appendix to improvement of Nakamula's upper bound for the absolute discriminant of a sextic number field with two real conjugates*, Econometric Inst., Erasmus Univ. Rotterdam, Report Series 9069/B, 1990.

ECONOMETRIC INSTITUTE, ERASMUS UNIVERSITY ROTTERDAM, P.O. BOX 1738, 3000 DR ROTTERDAM, THE NETHERLANDS

E-mail address: stroeker@cvx.eur.nl.bitnet